PRACTICE PROBLEMS FOR MIDTERM 2
MATH 430, SPRING 2014

Problem 1. *Let* $\mathfrak{A} = (\mathbb{R}; +, \cdot)$. *Give a formula in the language of* $\mathfrak{A}$ *which defines the following. (Here the language includes* $+$, $\cdot$, $\forall$, $\exists$, *variables, equality and logical connectives).*

(a) $\{0\}$
(b) $\{1\}$
(c) $\{3\}$
(d) the interval $(0, \infty)$
(e) $\{\langle r, s\rangle \mid r \leq s\}$ *(here* r, *and* s *are reals of course)*

Problem 2. *Write a deduction from* $\{\forall x Px, \forall x(Px \to Qx)\}$ *to* Qc. *(Here* c *is a constant symbol, and* P *and* Q *are one-place predicates). Note that you should not just prove that such a deduction exists, but actually write out the entire deduction.*

For the next problem you can use the theorem that if c is a constant symbol that does not appear in Γ, ϕ or ψ and $\Gamma \cup \{\phi^x_c\} \vdash \psi$, then $\Gamma \cup \{\exists x\phi\} \vdash \psi$ (Rule EI on page 124).

Problem 3. *Show that:*

(a) $\forall x(\alpha \to \beta) \vdash (\exists x\alpha \to \exists x\beta)$
(b) $\vdash \exists x(Py \wedge Qx) \leftrightarrow (Py \wedge \exists xQx)$.
(Hint: show that $\exists x(Py \wedge Qx) \vdash (Py \wedge \exists xQx)$ *and* $(Py \wedge \exists xQx) \vdash \exists x(Py \wedge Qx)$ *and then use the Deduction theorem and Rule T).*

Problem 4. *Show that if* x *does not occur free in any formula in* Γ, *then the set* $S = \{\phi \mid \Gamma \vdash \forall x\phi\}$ *is closed under modus ponens (i.e. whenever* ϕ_1 *and* $\phi_1 \to \phi_2$ *are both in* S, *then so is* ϕ_2*).*

Problem 5. *Suppose* Γ *is a consistent set of formulas. Show that* Γ *can be extended to a consistent* Δ *such that for every formula* ϕ, *either* $\phi \in \Delta$ *or* $\neg\phi \in \Delta$.

Problem 6. *Suppose* Γ *is a consistent set of formulas such that for every formula* ϕ, *either* $\phi \in \Gamma$ *or* $\neg\phi \in \Gamma$ *(i.e.* Γ *is complete). Define a relation* E *on the set of terms by*

$$t_1 E t_2 \text{ iff } \text{“}t_1 = t_2\text{”} \in \Gamma.$$

Prove that E *is an equivalence relation (i.e. that* E *is reflexive, symmetric, and transitive).*

Here you can use without proof that: $\vdash \forall x\forall y(x = y \to y = x)$ *and* $\vdash \forall x\forall y\forall z(x = y \to y = z \to x = z)$. *(Both can be obtained using generalizations and the equality axioms).*

Problem 7. *The soundness theorem says that:*
(a) If $\Gamma \vdash \phi$*, then* $\Gamma \models \phi$*.*
(b) If Γ *is satisfiable (i.e. some model* $\mathfrak{A} \models \Gamma$*), then* Γ *is consistent.*
Show that the two statements are equivalent.

Problem 8. *The completeness theorem says that:*
(a) If $\Gamma \models \phi$*, then* $\Gamma \vdash \phi$*.*
(b) If Γ *is consistent, then* Γ *is satisfiable.*
Show that the two statements are equivalent.

Problem 9. *The compactness theorem says that:*
(a) If $\Gamma \models \phi$*, then for some finite* $\Delta \subset \Gamma$*,* $\Delta \models \phi$*.*
(b) If every finite subset of Γ *is satisfiable, then so is* Γ*.*
(1) *Show that the two statements are equivalent.*
(2) *Prove the compactness from the completeness theorem*

Problem 10. (1) *Write a formula* λ *(in the language of equality) such that for every model* $\mathfrak{A}$*,* $\mathfrak{A} \models \lambda$ *iff the universe of* $\mathfrak{A}$ *has at least* 3 *elements.*
(2) *Show that there is a set of sentences* Σ *such that for every model* $\mathfrak{A}$*,* $\mathfrak{A} \models \lambda$ *iff* $\mathfrak{A}$ *is infinite.*
(3) *Let* Γ *be a set of formulas and suppose* Γ *has arbitrarily large models. I.e. for every* k *there is a model with at least* k *elements that satisfies* Γ*. Use compactness to show that* Γ *has an infinite model.*

Problem 11. (1) *Show that for any model* $\mathfrak{A}$*,* $Th(\mathfrak{A})$ *is complete.*
(2) *Let* $\mathcal{K}$ *be the collection of all groups. Show that* $Th(\mathcal{K})$ *is not complete.*

Problem 12. *Let* $\mathfrak{A} = (\mathbb{N}, 0, S, <)$*. Construct a countable model* $\mathfrak{B}$*, such that* $\mathfrak{A} \equiv \mathfrak{B}$*, but* $\mathfrak{A} \not\cong \mathfrak{B}$*.*